Math 120A
Differential Geometry

Sample Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a smooth curve. Define a reparametrization $\widetilde{\gamma}$ of γ.

(b) [5pts.] Show that reparametrization is transitive: if $\widetilde{\gamma}$ is a reparametrization of γ, and $\widehat{\gamma}$ is a reparametrization of $\widetilde{\gamma}$, then $\widehat{\gamma}$ is a reparametrization of γ.

Problem 2.

(a) [5pts.] Give a formula for the curvature of (i) a unit speed curve and (ii) an arbitrary regular curve in $\mathbb{R}^3$.

(b) [5pts.] Let $\gamma(s)$ be a unit speed plane curve with signed curvature function $k(s)$. Let $\gamma_a(s) = a\gamma(s)$ be the image of γ under the dilation $\mathbf{v} \to a\mathbf{v}$, for a a positive constant. Prove that the signed curvature of γ_a (expressed in terms of *its* arclength) is $\frac{1}{a}k(\frac{s}{a})$.

Problem 3.

(a) [5pts.] Give a formula for the torsion of (i) a unit speed curve in $\mathbb{R}^3$ and (ii) a regular curve in $\mathbb{R}^3$. Do not forget to mention any hypotheses you need for your formula to make sense.

(b) [5pts.] Show that if γ is a regular curve with $\tau \equiv 0$ (and defined everywhere), then γ lies in a plane. [Hint: Recall that the equation of a plane is $\mathbf{x} \cdot \mathbf{v} = d$, where $\mathbf{v}$ is a constant vector and d is a scalar. What vector do you expect $\mathbf{v}$ to be?]

Problem 4.

Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a regular curve.

(a) [5pts.] Define the arclength of $\gamma(t)$.

(b) [5pts.] Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a regular T-periodic curve, and let ℓ be the length of the curve over $[0, T]$. Prove that a unit-speed reparametrization of γ is ℓ-periodic.

Problem 5.

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a unit-speed curve of nowhere-vanishing curvature, and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the orthonormal system consisting of the unit tangent, normal, and binormal vector functions. Suppose that $\mathbf{t}$ makes a fixed angle with a constant unit vector $\mathbf{a}$, i.e. $\mathbf{t} \cdot \mathbf{a} = \cos\theta$, for θ constant.

(a) [3pts.] Prove that $\mathbf{n} \cdot \mathbf{a} = 0$.

(b) [3pts.] Prove that $\mathbf{a}$ can be written as a linear combination $\mathbf{a} = \lambda\mathbf{t} + \mu\mathbf{b}$. What are the coefficients? (Keep in mind that $\mathbf{a}$ is a unit vector.)

(c) [4pts.] Show that if κ and τ are the torsion and curvature of γ, then $\tau = \pm\kappa\cot\theta$. [Hint: What is the derivative of the relationship you found in part (b)?]